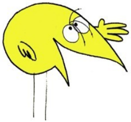
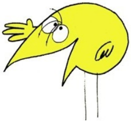
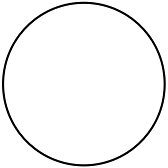
Problem 3: Coloured Circles

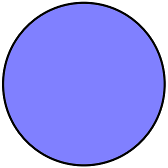


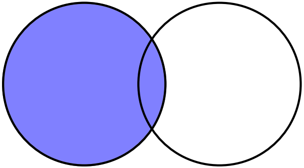


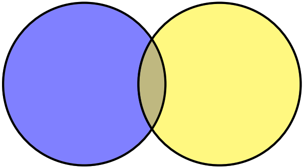
France 1
Henry Bambury

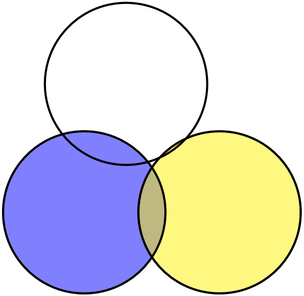
ITYM
2015

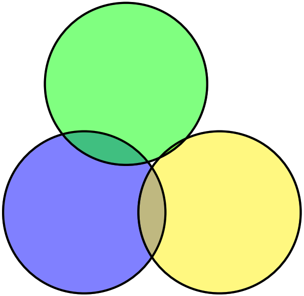


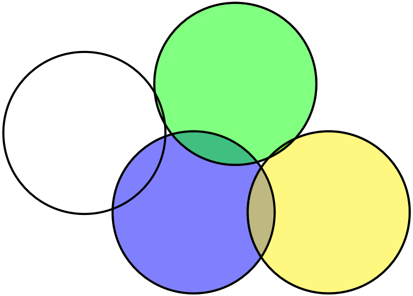


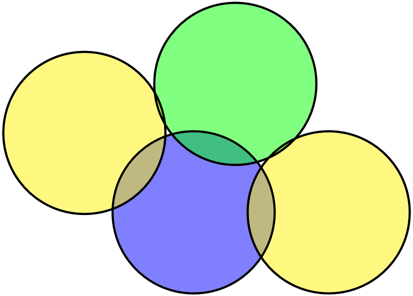


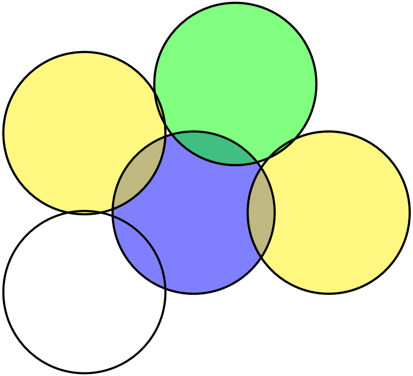


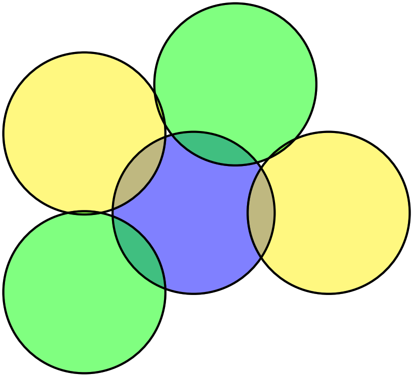


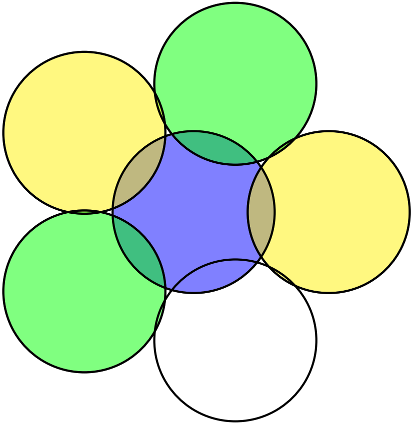


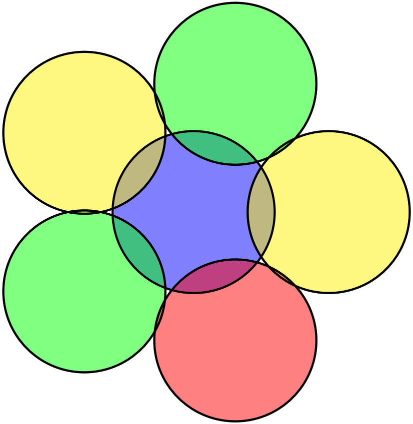












Notations

For all lower bound, we suppose that n is big enough.

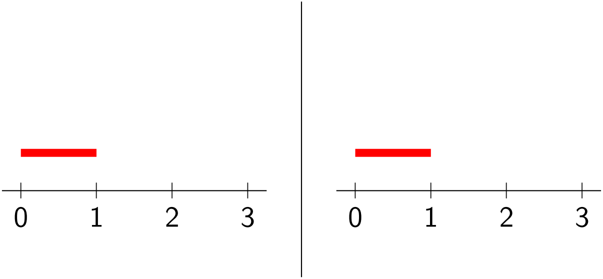
$n \rightarrow$	# balls
$d \rightarrow$	dimension
$k \rightarrow$	each point of the plane is covered by at most k balls
$\mathcal{C}_d(n,k) \rightarrow$	minimal # of colors needed to colour the balls, when Alice plays as she wishes
$\mathcal{C}_d^c(n,k) \rightarrow$	all balls have the same radius

Lemma

$$\mathcal{C}_1^c(n, k) \leq 2k - 1$$

Lemma

$$\mathcal{C}_1(n, k) \geq k + 1$$



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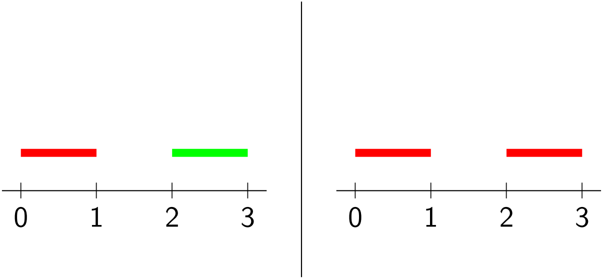

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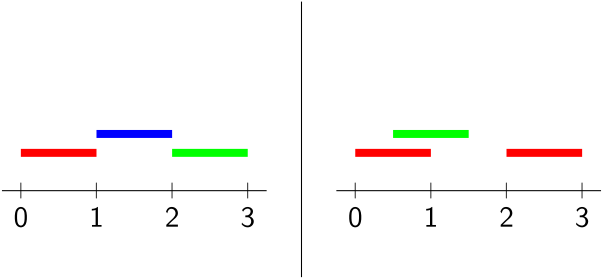

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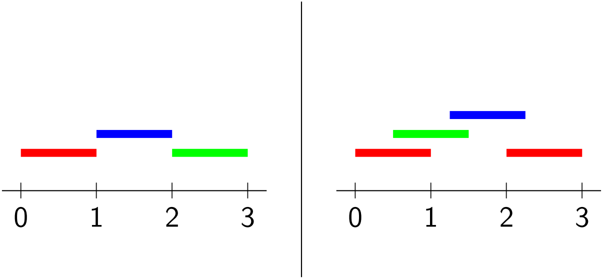

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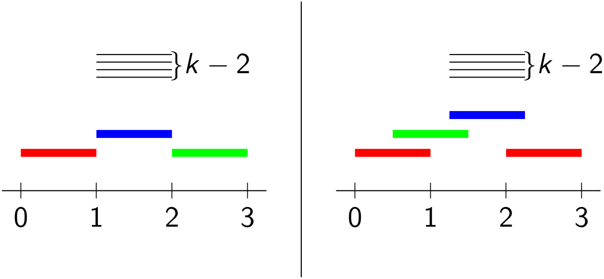
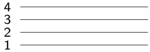


Lemma

$$\mathcal{C}_1(n, k) \geq \frac{5k-1}{4}$$

$$\boxed{k = 9}$$



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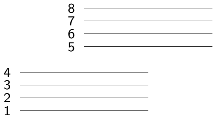

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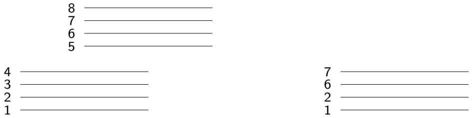

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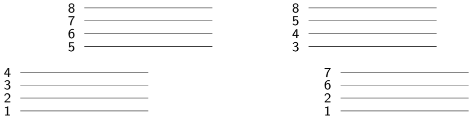

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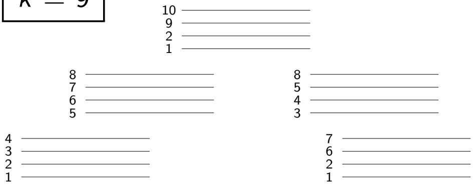

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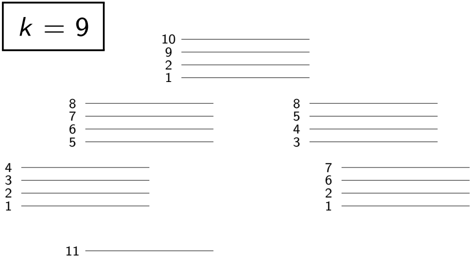
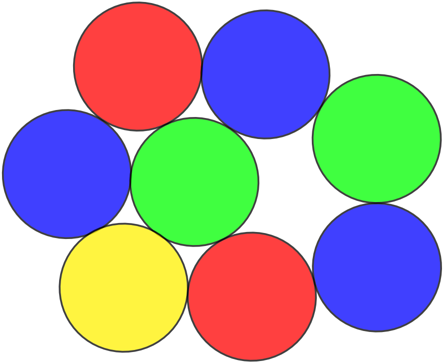


Lemma

$$\mathcal{C}_2(n, 2) \geq 4$$

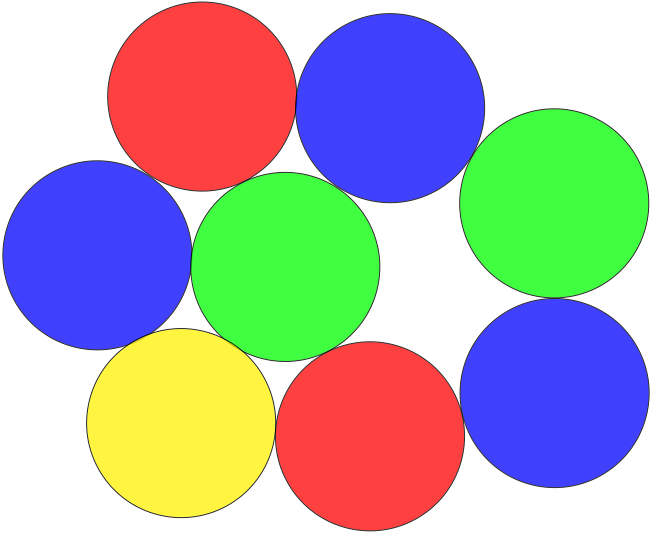


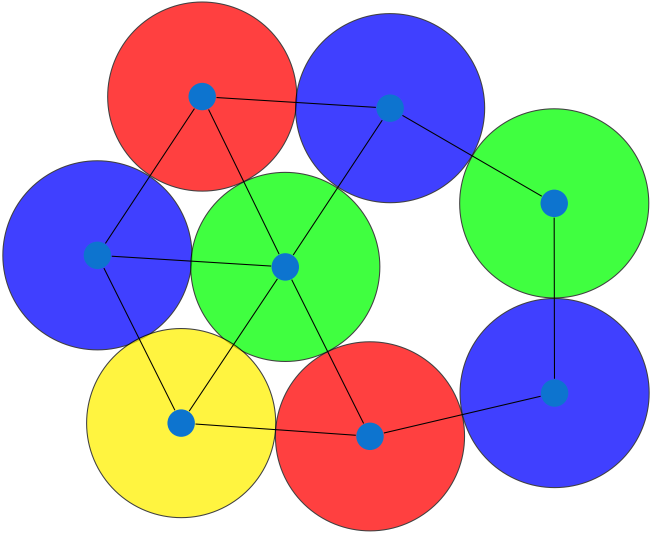
Four-colors theorem

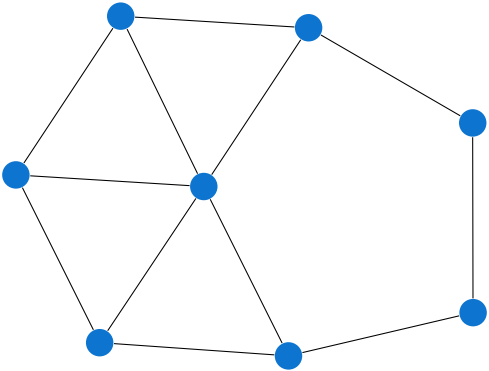
The vertices of every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color.

$$\begin{cases} \mathcal{C}_2(n,2) \geq 4 \\ \mathcal{C}_2(n,2) \leq 4 \end{cases} \Rightarrow \mathcal{C}_2(n,2) = 4^*$$

*Supposing that Alice places all her circles, and then Bob colours them

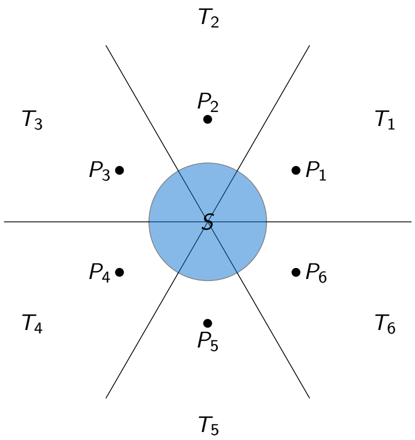






Theorem

One disc intersects at most $7k - 1$ other discs. *(Generalization of shortlist IMO 2003)*


T_2
T_3
P_2
T_1
P_3
P_1
S
P_4
P_6
T_4
P_5
T_6
T_5

Theorem

One disc intersects at most $7k - 1$ other discs. *(Generalization of shortlist IMO 2003)*

Corollary

$$\mathcal{C}_2^c(n, k) \leq 7k$$

Complete graph for $n = 7$

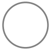
Propostion

$$\mathcal{C}_2(n, k) \geq \frac{7k}{3}$$

A *complete graph* is a graph in which every pair of distinct vertices is connected by a unique edge.

Complete graph for $n = 7$

$k = 3$



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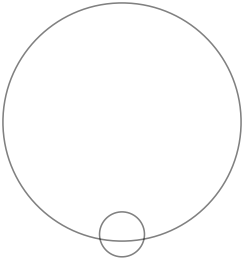

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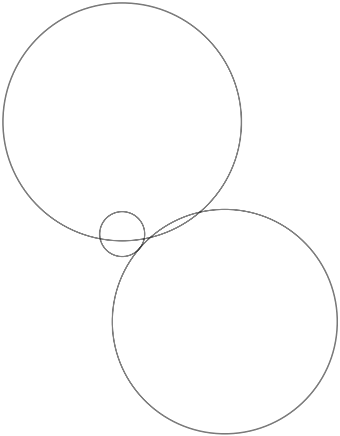

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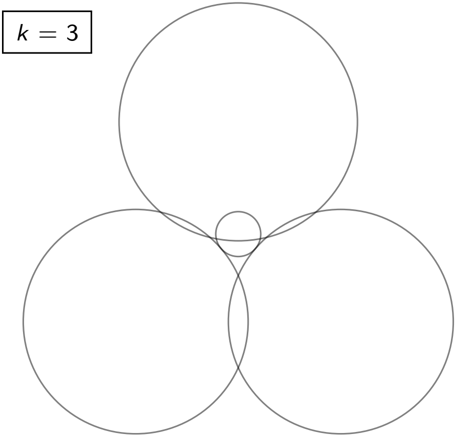

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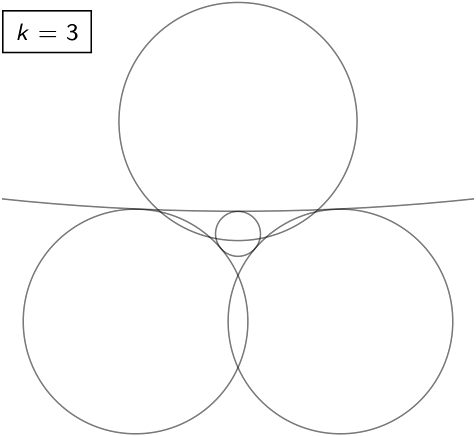

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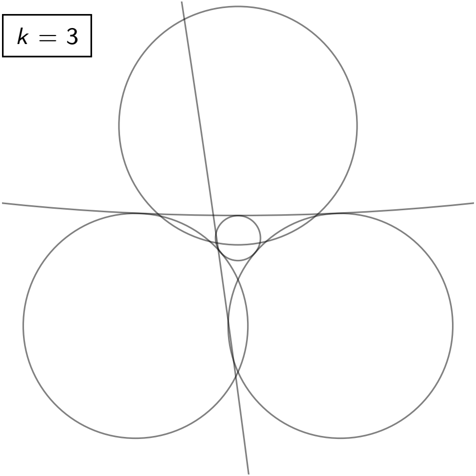

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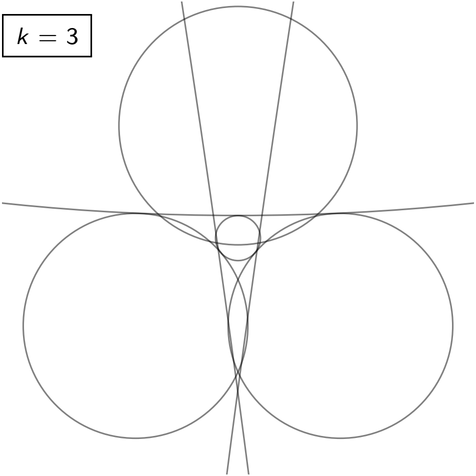
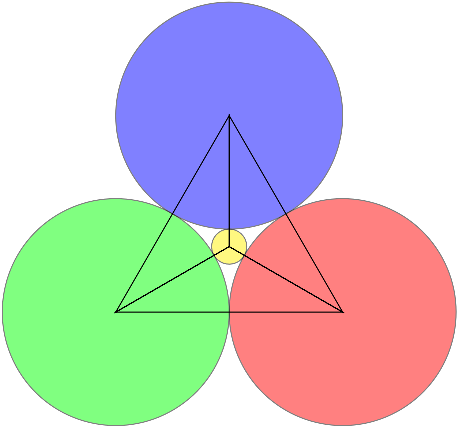


Lemma

$$\mathcal{C}_d(n, k) \geq k + d$$

Lemma

$$\mathcal{C}_d(n, k) \geq \frac{k}{2}(d + 2)$$



A *simplex* is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions

Lemma

$$\mathcal{C}_d^c(n, k) \leq 3^d k$$

Lemma

$$\lim_{d \to +\infty} \mathcal{C}_d(n, k) = n$$

Summary

$$\begin{aligned} \mathcal{C}_1^C(n,k) &\leq 2k-1 \\ \mathcal{C}_2^C(n,k) &\leq 7k \\ \mathcal{C}_d^C(n,k) &\leq 3^d k \end{aligned}$$

$$\begin{aligned} k+1 &\leq \mathcal{C}_1(n,k) \\ k+2 &\leq \mathcal{C}_2(n,k) \\ k+d &\leq \mathcal{C}_d(n,k) \end{aligned}$$

$$\begin{aligned} \tfrac{5k-1}{4} &\leq \mathcal{C}_1(n,k) \\ \tfrac{7k}{3} &\leq \mathcal{C}_2(n,k) \\ \tfrac{k}{2}(d+2) &\leq \mathcal{C}_d(n,k) \end{aligned}$$

Thank you for your attention ☺ *!*