Proof of *Nārāyaṇa* 's algorithm

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Here we prove that the algorithm given in *Gaṇitakaumudī* [1] , GK 14.10, 14.11-a indeed generates all 4×4 pandiagonal magic squares. The most frequent magic squares are those filled with the natural numbers $1, 2, \cdots, 16$. However the text does contain examples of squares containing non-consecutive integers, more precisely those with 4 arithmetic progressions, $P_1, \cdots, P_4$ where $P_t = \{a_t + kd, k = 0, 1, 2, 3\}$ with a_t as the first term and d the common difference and the condition that $a_1 + a_4 = a_2 + a_3$. There are also occurrences of the form $\{a_i, a_i + d_1, a_i + d_1 + d_2, a_i + 2d_1 + d_2\}$, for instance GK.14.15p, which are called generalised arithmetic progressions in contemporary language. The generic series is the one with $a_1 = 1, a_2 = 5, a_3 = 9, a_4 = 13, d_1 = d_2 = 1$ and is enough to understand the construction . The cells are filled with consecutive elements of the progressions using well-defined movements . While filling the square it is to be understood that the square folds on itself like a torus, i.e. all placements are modulo 4. We have numbered the cells as that of a matrix.

The possible movements from the $(i, j)^{th}$ cell are :

1. The movement, termed *turaga-gati*, of a *horse* as in chess, which is a displacement of two columns (rows) and one row (column) reaches the position $(i \pm 1, j + 2)$ or $(i + 2, j \pm 1)$. The first two are called *descending* and *ascending* horse movements and the last two are named *right* and *left*. We will use the letters D, A, R and L for them.

2. A *diagonal* move that consists of a displacement of one place on both the row and column reaching $(i \pm 1, j \pm 1)$. The directions descending-right, .., ascending-left will be denoted by δ_ρ (or ρ_δ), $\delta_\lambda, \alpha_\rho, \alpha_\lambda$.

3. An *adjacent* move is one step exclusively on a row (or column) and noted by a, d, r, l.

4. A *vertical* (respectively *horizontal*) move is two steps along a row (respectively column).

5. An *antipodal* move comprises two steps on one of the eight diagonals ending at the $(i + 2, j + 2)^{th}$ cell. Two such diametrically opposed elements are called *antipodes*.

The movements 1, 2 and 4 are mentioned explicitly in GK.14.10. The term *antipodal* is used by Rosser and Walker [2] and can be identified in arabic texts on magic squares as *fīl* (to designate the bishop of chess).

Each of the four progressions P_t , termed *pada* for step, is placed in the square using a sequence of three movements horse-diagonal-horse containing two identical horse movements. The whole step has what we will call, a *sense* that is identical to that of the horse movement since the intervening diagonal follows the same direction as that of the horse. Thus the sequence of moves $L\alpha_\lambda L$ and $L\delta_\lambda L$ are both of the left sense while the sense of the sequence $A\lambda_\alpha A$ is ascending. The first and second and again third and fourth progression of numbers are linked by an adjacent move while between P_2 and P_3 the movement is antipodal. We notice that the constructions involving the horizontal and vertical movements can also be seen as horse-diagonal-horse movements by considering a different arrangement of the progressions, for example, for the generic case with $d = 2, a_1 = 1, a_2 = 2, a_3 = 9, a_4 = 10$.

Corresponding to a movement from any (i, j) to $(i+m, j+n)$ there exists an unique *inverse* movement that takes any (i, j) to $(i-m, j-n)$. For example the right horse and the left horse are inverse movements of each other and when applied twice successively come back to the starting point. Similarly the δ_ρ diagonal move has α_λ as its inverse.

The constant sum of the elements of rows, columns and the 8 diagonals (2 principal and 6 broken), called the *magic sum* is denoted by S.

Theorem 1. Nārāyaṇa Paṇḍita *'s algorithm generates* 4×4 *pandiagonal magic squares.*

Proof. The elements of any progression are placed using the sequence of movements horse-diagonal-horse, hence they are all on different rows and columns . Depending on the sense of this sequence of moves, there is a shift of three rows (or columns) and one column (respectively row) between the beginning and end of the step. For instance beginning at (i, j) using two right horse movements one would reach $((i+2)+2, (j+1)+1) = (i, j+3)$ The diagonal move employed would be necessarily to the right . So the cell finally reached would be $(i \pm 1, j + 3)$ depending on whether the diagonal move is δ_ρ or α_ρ . In the same way any chosen sense starting at (i, j) ends diagonally at $(i \pm 1, j \pm 1)$. In particular for the generic case, 4 is *diagonal* to 1. By construction 5 is placed adjacent to 4 but on a row and column different from 1. Therefore the adjacency can not be again in the sense of the placement of P_1. Thus the $R\delta_\rho R$ moves above could be followed by placing 5 with a left movement bringing it to $(i + 1, j + 2)$. Now the placement of P_2 using $R\alpha_\rho R$ brings the element 8 to $(i, j+1)$. A down movement to place 5 would terminate at $(i+2, j-1)$. This time P_2 following $L\delta_\lambda L$ brings the element 8 to or $(i-1, j)$. Thus by construction 5,6,7 and 8 which are themselves on different rows and columns are also on different rows and columns from 1,2,3 and 4 respectively and 8 is always *adjacent* to 1. Now 9 which is antipodal to 8 by construction begins at a row and column different from both 1 and 5 and 13 does the same. Thus each row and column of the square contains exactly one element of a different position of the four progressions . This constant sum is $S = a_1 + a_2 + a_3 + a_4 + d(0 + 1 + 2 + 3) = 2(a_1 + a_4) + 6d$ and for example for the generic case we get the desired $S = 34$.

Since the movements used to place P_3 and P_4 are inverses of P_2 and P_1 respectively and P_2 and P_3 are connected by an antipodal movement the elements a_t+kd and $a_{5-t}+(3-k)d$ for $k = 0, 1, 2, 3$ and $t = 1, 2, 3, 4$ are in antipodal positions. So a pair of antipodes sum up to $a_t + a_{5-t} + 3d$ which is exactly $S/2$.

Since any diagonal, main or broken, contains two pairs of antipodal elements, the sum of its elements is S. □

To show that this algorithm generate all pan-diagonal magic squares we use the two following properties .

Lemma 1. *In a 4×4 pan-diagonal magic square the antipodal elements always add up to half the magic sum.*

Proof. Other than being part of a row and a column every element shares the two diagonals containing it with its antipode. Thus there are six distinct cases involving at least one of a pair of antipodal elements where the constant sum of the magic square is attained with the antipodal elements each occurring at the intersection of four cases. The other 18 summands of these six cases include another pair of antipodal elements which are again constituants of a pair of common diagonals which use up 10 summands. Now there are two more diagonals which exactly cover all elements of the initial six cases. The initial sum of elements is $6S$ and if we substract $4S$, the sum of the elements in the four diagonals above, only the antipodal elements remain with a multiplicity of 4.

In Figure 1 there are six blue lines passing through x or y . The other cells covered by the blue lines are taken care of by the four green diagonals. The simultaneous presence of a green and a blue line in a cell annules the element of the cell and only the antipodes x and y remain at the intersection of 4 blue lines each. Thus $6S - 4S = 2S = 4(x + y)$.

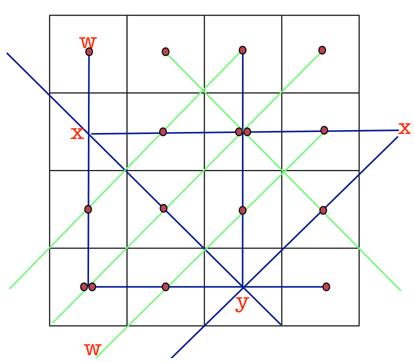

Figure 1: Sum of Antipodes

□

Lemma 2. *In a 4×4 pan-diagonal magic square the sum of elements contained in 2×2 squares obtained with adjacent cells is the magic sum . This is also true for such 2×2 squares obtained by folding.*

Proof. The two rows and columns containing the elements that form a sub-square of adjacent elements also contain elements that belong to exactly two diagonals. The sum of the latter is $2S$ which subtracted from the former, $4S$, leaves twice the sum of elements of the 2×2 square.

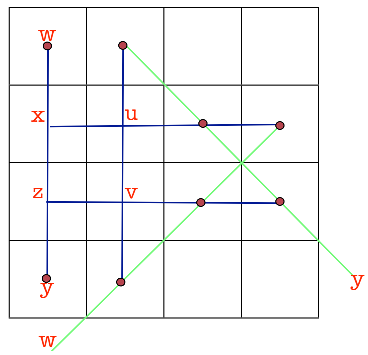

Figure 2: Sum of 2×2 squares

In the above example $2(x + u + z + v) = 4S - 2S = 2S$ □

We denote by x' the antipode of x. It is sometimes convenient to use two sets $\mathcal{I} = \{x_1, \cdots, x_8\}$ with the initial 8 elements that fill the magic square and $\mathcal{A} = \{x'_1, \cdots, x'_8\}$ the set of their antipodes.

Corollary 1. *In a 4×4 pan-diagonal magic square two elements can be placed diagonally, vertically or horizontally only if their sum can be expressed as the sum of two admissible elements of the magic square in at least two different ways. For elements to be adjacent at least four different representations must exist.*

Proof. For any 4 elements, say x, y, z', w' that form a 2×2 square of adjacent cells their antipodes also form a similar square (Figure 3). Now if x, w' are the diagonal elements of the first square, then y', z are diagonals of the second one and the sums are identical since $x + w' = S - (y + z') = S/2 - y + S/2 - z' = y' + z$. Similarly if x, u are vertical then the antipodes of the other two elements of the column s, z are also vertictal elements and have the same sum.

A pair of adjacent elements , say in a row, belong simultaneously to two 2×2 overlapping squares and one row. When their sum is removed from S we get three representations for the sum of the other pair of adjacent elements present in the row. This last pair is again present in two overlapping squares which gives a fourth representation of the same sum. □

Corollary 2. *Every row, every column and every 2×2 square of adjacent cells of a pan diagonal magic square contains exactly two elements of $\mathcal{I}$ and two from $\mathcal{A}$. If these elements are x, y, w', z' then $x + y = w + z$.*

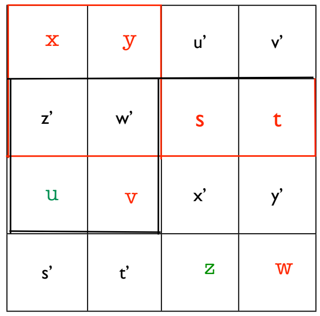

Figure 3

Proof. Clearly four elements all chosen from $\mathcal{I}$ or $\mathcal{A}$ do not give the magic sum. If a row or column is filled with exactly three elements of $\mathcal{I}$ we are left with equalities of the form $a + b' = c + d = e + f = g + h$ with all summands but b' belonging to $\mathcal{I}$. Now $a + b' \geq 10$ and since this sum must also be split in three distinct ways in $\mathcal{I}$, $c + d \leq 11$. Now since $10 \leq a + b' \leq 11$ the only choices of b are 8 and 9 both of which occur in the 3 decompositions of 10 or 11. Thus exactly two elements from each set are necessary for filling a row, column or 2×2 square of adjacent elements. $\square$

In fact up to permutations of rows and columns a generic pan-diagonal magic square is of the type

a	b	c'	d'
e'	f'	g	h
c	d	a'	b'
g'	h'	e	f

with $a + b = g + h = c + d = e + f = 9$ and $a + e' = d + h'$.

Theorem 2. Nārāyaṇa Paṇḍita *'s algorithm generates all* 4×4 *pan-diagonal magic squares.*

Proof. Let us place 1 at the top left hand corner. Now 2 can not be diagonal,vertical or adjacent by Corollary 1.Therefore 2 is at a horse movement away from 1. By the same argument 1 and 3 are horse positions away. The possible horse positions from (i,j) are $(i \pm 1, j + 2), (i + 2, j \pm 1)$. Thus if 2 is at one of these positions , say $(i + 1, j + 2)$ then 3 is either diagonal from 2 , i.e. $(i + 2, j \pm 1)$ or at a vertical move away (i-1, j+2).

1			
		2	
	3		3
		3	

These are the two positions mentioned explicitly as *aikya ekāntara*. Once 1, 2 and 3 are fixed, their antipodes are as well and hence the only possibility for the square containing 2 and 3, if they are diagonal, are 16 and 13. The antipode of 13, i.e. 4, is thus at the same horse movement from 3 as 2 is from 1, i.e. at $(i+2+1, j+2+1) = (i-1, j-1)$ which is on a diagonal containing 1. Continuing with the above example and choosing 3 to be the left diagonal movement,

1			
	$4'$	2	
	3	$1'$	
			4

In case 2 and 3 are vertical/horizontal this column/row must contain 13 and hence the position of 4 is determined uniquely. We do not treat this case in detail because it can as well be considered as a horse-diagonal-horse movement by using another progression $\{1, 3, 5, 7\}$. . Now 1,2,3 and 4 are all on different rows and columns. The next element in the sequence 5 can not be reached from 4 by a horse movement since the only available places are adjacent to 1 and this is not possible by Corollary 1. If 4 and 5 were diagonal, vertical or horizontal, 5 would be adjacent to 2 or 3 which would similarly be incoherent. Hence 5 takes up one of the available cells adjacent to 4.

Once five elements and therefore their antipodes are fixed, three elements of four unfilled lines are known (5 is placed to the left of 4 in our example) and therefore the fourth is determined uniquely.

1			$3'$
$5'$	$4'$	2	
	3	$1'$	
$2'$		5	4

Even without actually working out the missing fourth element it can be seen for instance that in the above example 6 which is never adjacent to 1 or 2 can further not fill a 2×2 square already containing two elements of $\mathcal{I}$ (Corollary 2) and must only be at that horse position from 5 as 4 is from 3. The same argument places 7 diagonally up and left from 6 and 8 follows by a up-horse movement. Now its antipode is fixed. We notice that the antipodes must indeed follow the inverse movements, for if x at the $(i, j)^{th}$ position is moved to y at $(i+m, j+n)$ then y' of the $(i+m+2, j+n+2)^{th}$ cell by the inverse movement goes to $(i+m+2-m, j+n+2-n)$, i.e. x'.

□

Remark 1. *In contemporary language we would say that the group of* 4×4 *pan-diagonal magic squares is isomorphic to the Coxeter-Weyl group of* $W(B_4)$.

References

[1] Dvivedī, P. ed. (1942) The *Gaṇitakaumudī* of *Nārāyaṇa Paṇḍita* Part II, Saraswati Bhavana Texts 57, Benares, 1942 .

[2] Rosser, J. B. and Walker, R. J. , On the Transformation Group for Diabolic Magic Squares of Order Four, *Bull. Amer. Math. Soc.* Volume **44** Number 6 (1938), 416–420.